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Counting supertubes

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Abstract: The quantum states of the supertube are counted by directly quantizing the linearized Born-Infeld action near the round tube. The result is an entropy $S = 2\pi\sqrt{2(Q_{D0}Q_{F1} - J)}$, in accord with conjectures in the literature. As a result, supertubes may be the generic D0-F1 bound state. Our approach also shows directly that supertubes are marginal bound states with a discrete spectrum. We also discuss the relation to recent suggestions of Mathur et al involving three-charge black holes.

Keywords: D-branes, Black Holes in String Theory.

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1. Introduction

Two hallmarks of modern stringy physics are the non-abelian interactions associated with multiple branes, and the idea that such non-abelian behavior can simplify and dramatically change form in the large N limit. Gauge/gravity dualities [1]–[4] and the related stringy counting of black hole entropy [5] are examples where there seems to be a strict duality at large N. However, many other interesting examples arise in brane polarization effects (see e.g. [6, 7]) where a bound state of low-dimension branes may, when polarized, be effectively described as a single brane of higher dimension.

We will be concerned here with the D0-F1 supertube of [8]–[10], which is an example of such brane polarization. Supertubes have the special distinction that the polarized states are BPS and arise without the application of an external field. Mathur et al. [11]–[14] have suggested that supertubes are also connected to black hole entropy (and thus to AdS_3/CFT_2).

The supertubes of interest here carry D0 and F1 charge and have the supersymmetries expected of such configurations. These are the original supertubes of [8], though many related configurations can be obtained through duality transformations. The charges are arrayed around a tube of topology $S^1 \times \mathbb{R}$ in space, where the $\mathbb{R}$ represents a translation symmetry of the system and the direction along which the fundamental strings are aligned. Interestingly, the S^1 can be an arbitrary curve [15] (see also [16]–[19] for earlier and related results) in the space of symmetry orbits; all such configurations are static. We assume the curve is closed and also compactify the $\mathbb{R}$ direction as we are interested in cases with finite charge.

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Mateos and Townsend [8] showed that the supertube can be described using the Dirac-Born-Infeld effective action of a D2-brane. The D2-worldvolume is then the above-mentioned $S^1 \times \mathbb{R}$ tube ($\times$ time). Because the S^1 is a closed curve, the configuration has no net D2 charge. However, if the U(1) electric and magnetic fields (E and B) are switched on, the configuration gains both a net D0 and a net F1 charge. Supertubes arise when the electric field reaches $E = 1$ in string units (with $2\pi\alpha' = 1$) and when B is nowhere vanishing. The static nature of the supertube can be understood as a balance between the D2-brane tension and the Poynting angular momentum from the simultaneous presence of both electric and magnetic fields [8, 15].

It is natural to conjecture [8] that supertubes describe D0-F1 bound states. Because they would be marginal such states, it is nontrivial to verify that they are in fact bound. However, we will demonstrate this in section 3 through an explicit quantization of the system in which the spectrum of BPS states is shown to be discrete[1].

A much stronger conjecture is that *almost all* D0-F1 bound states are supertubes for large Q_{D0}, Q_{F1}, J. This would be of great interest, as supertubes would then provide a *geometric* description of these bound states; states of the supertube are directly labelled by the shape of the S^1 cross-section and by the magnetic field as a function of location on this S^1. In contrast, previously known descriptions are highly non-geometric; the states are only indirectly understood in terms of the non-abelian D0-F1 gauge theory. Mathur et al have provided evidence that this conjecture is correct, but an explicit quantization and counting has remained lacking. Mathur et al have also made interesting further conjectures extrapolating to the three-charge systems associated with black holes, but we will save discussion of such conjectures for section 4.

An intermediate conjecture was made in [21] to the effect that supertubes are in fact a significant fraction of such BPS D0-F1 bound states. In particular, Lunin, Maldacena, and Maoz conjectured that the entropy of a supertube configuration with Q_{D0} units of D0 charge and Q_{F1} units of F1 charge is $S = 2\pi\sqrt{2Q_{D0}Q_{F1}}$ to leading order in large charges. Note that this is just the leading order expression for the entropy of all BPS D0-F1 bound states, which can be computed from the fact that the system is dual to the fundamental string with right-moving momentum, whose entropy is in turn given by the Cardy formula [22]. It is also of interest to count supertubes with fixed angular momentum J, in which case the corresponding conjecture would be $S = 2\pi\sqrt{2(Q_{D0}Q_{F1} - J)}$ [23] . The main point of our work below is to verify this conjecture by using a linearization of the D2-brane effective action to directly count quantum supertube states.

The structure of this paper is as follows. We begin with some preliminaries in section 2, gauge fixing the D2 effective action and linearizing about the round supertube configuration (in which the S^1 is a circle). However, we save the general justification of certain interesting relations for appendix A and the detailed form of certain expressions for appendix B. The spectrum of states is then computed in section 3, whence it is straightforward to count

[1] We will use a linearized description in which the DeWit-Hoppe-Nicolai continuum of membrane states [20] does not arise. This is consistent [1] with our intent to study a single bound state, and not the second-quantized theory of supertubes.

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the states and to establish that our results are valid when $Q_{D0}Q_{F1} - J \ll Q_{D0}Q_{F1}$ and $Q_{D0}Q_{F1} - J \gg 1$. As stated above, our counting verifies that supertubes are marginal bound states with an entropy given by $S = 2\pi\sqrt{2(Q_{D0}Q_{F1} - J)}$.

Finally, we close in section 4 with a summary and a discussion of the implications for the further conjectures of Mathur et al. [11]–[14] relating to three-charge black holes. While the D2 action used here is not sufficient to describe the relevant three-charge systems,[2] one expects the three-charge calculations to proceed along similar lines.

2. Preliminaries: gauge fixing and linearization

As indicated in the introduction, our starting point will be the D2 Born-Infeld effective action:

$$S_{D2} = -T_{D2}\int d^3\xi\sqrt{-\det(g+\mathcal{F}))} - T_{D2}\int C_{[1]}\wedge\mathcal{F} + \text{Fermions}\,, \tag{2.1}$$

where $\mathcal{F}_{\mu\nu} = F_{\mu\nu} + B_{\mu\nu}$ and we have included the Chern-Simons term representing the coupling to a background Ramond-Ramond vector potential $C_{[1]}$ and a Neveu-Schwarz two-form potential $B_{\mu\nu}$, though we will soon set all background fields to zero. T_{D2} is the D2-brane tension. It will not be necessary to treat the Fermions explicitly since the configurations of interest preserve 8 supercharges [8]; thus the unbroken supercharges may be used to deduce the behavior of the Fermions from that of their bosonic cousins.

Our tasks are to isolate the physical degrees of freedom and to find a description in which the states can be counted. To this end we impose a version of static gauge and then expand the action and all relevant quantities to quadratic order in fields, taking the round supertube (for which the S^1 is an isometry direction) as the base point of the expansion. Note [9, 15] that this is the unique configuration saturating the angular momentum bound on supertubes. As a result, it will have certain nice properties reminiscent of vacuum states.

We now set all closed string fields to zero, e.g., we take the supertube to be embedded in Minkowski space, and we set $B_{\mu\nu} = 0$ and $C_{[1]} = 0$. It will be convenient to rename the worldvolume coordinates $\xi^0 = t$, $\xi^1 = \sigma$, $\xi^2 = z$. Taking X^μ to be cartesian coordinates on Minkowski space, our static gauge is then defined by the conditions $t = X^0$, $z = X^3$, and $\tan\sigma = X^1/X^2$. Thus, z represents the coordinate along the length of the tube, while σ is an angular coordinate around each S^1 cross-section of the tube. It is also convenient to introduce the radius $R(t,z,\sigma)$ in the X^1X^2 plane defined by $R^2 = (X^1)^2 + (X^2)^2$ and the notation $E = F_{tz}$, $B = F_{z\sigma}$. Finally, we choose an electromagnetic gauge in which the world-volume vector potential A satisfies $A_t = 0$. In particular, we take $A = Etdz + Bzd\sigma$ for the background supertube.

In the usual way, the action (2.1) shows that the D2 brane carries both D0 and F1 charge (in the z direction). We are most interested in static D2 branes invariant under z-translations, as adding time dependence or z-momentum breaks all supersymmetries. The

[2] See [24] for a Born-Infeld discussion of three-charge supertubes.

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charges for this special case are given ([15]) by

$$Q_{D0} = \frac{T_{D2}}{T_{D0}} \int dz d\sigma B \,, \tag{2.2}$$

$$Q_{F1} = \frac{1}{T_{F1}} \int d\sigma \Pi_z = \frac{T_{D2}}{T_{F1}} \int d\sigma \frac{E|\partial_\sigma X|^2}{\sqrt{(1-E^2)|\partial_\sigma X|^2 + B^2}} \,, \tag{2.3}$$

where T_{D0}, T_{F1} represent the tensions of the appropriate branes and we have normalized Q_{D0}, Q_{F1} so that they take integer values. The angular momentum J (in the X^1X^2 plane) and energy P^0 take the form

$$J = T_{D2} \int d\sigma \frac{EB(X_1 \partial_\sigma X_2 - X_2 \partial_\sigma X_1)}{\sqrt{(1-E^2)|\partial_\sigma X|^2 + B^2}} \,, \tag{2.4}$$

$$P^0 = \int dz d\sigma \mathcal{T}^{00}(X(\xi)) = T_{D2} \int dz d\sigma \frac{B^2 + |\partial_\sigma X|^2}{\sqrt{(1-E^2)|\partial_\sigma X|^2 + B^2}} \,, \tag{2.5}$$

where $\mathcal{T}^{00} = \frac{2}{\sqrt{-G}} \frac{\partial L}{\partial G_{00}}$ is the stress-energy tensor on the D2-brane. The energy and charges satisfy a BPS bound [15] of the form

$$P^0 \geq T_{D0}|Q_{D0}| + L_z T_{F1} |Q_{F1}| \,. \tag{2.6}$$

We emphasize that the above charges are defined by the coupling of the D2 action (2.1) to the metric and external gauge fields.

The round supertube is then given [8] by

$$R_{\text{round}}(t, z, \sigma) = R \,, \tag{2.7}$$

$$X^i_{\text{round}}(t, z, \sigma) = 0 \qquad \text{for} \qquad i = 4, 5, 6, 7, 8, 9 \,, \tag{2.8}$$

$$(F_{tz})_{\text{round}} = \pm 1 \,, \qquad (F_{z\sigma})_{\text{round}} = B \,, \tag{2.9}$$

where, from here on, R and B are constants that determine the charges and angular momentum of the round supertube about which we expand below.

In order to make these charges finite, let us periodically identify the system under $z \to z + L_z$, so that the supertube at any time is an $S^1 \times S^1$ embedded in $S^1 \times \mathbb{R}^9$. The total charges and angular momentum of the round tube are then

$$Q^{\text{round}}_{D0} = \frac{2\pi L_z T_{D2}}{T_{D0}} B \,, \tag{2.10}$$

$$Q^{\text{round}}_{F1} = \text{sgn}(EB) \frac{2\pi T_{D2}}{T_{F1}} \frac{R^2}{B} \,, \tag{2.11}$$

$$J^{\text{round}} = \text{sgn}(EB) 2\pi L_z T_{D2} R^2 \,. \tag{2.12}$$

Note in particular that since $T_{D0}T_{F1} = 2\pi T_{D2}$ (see, e.g. [25]), we have $J^{\text{round}} = Q^{\text{round}}_{D0} \times Q^{\text{round}}_{F1}$.

Our final task is to expand the action (2.1) to quadratic order about the round supertube solution. Let us denote the deviations from the round solution by

$$R = R_{\text{round}} + r, \; X^i = X^i_{\text{round}} + \eta^i \,, \tag{2.13}$$

$$A = A_{\text{round}} + a, \; F_{tz} = E_{\text{round}} + e_z \,, \qquad F_{z\sigma} = B_{\text{round}} + b \qquad \text{and} \qquad F_{t\sigma} = e_\sigma \,. \tag{2.14}$$

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It is then straightforward but tedious to expand the quantities of interest to quadratic order in η, a. Note that we also wish to compute the hamiltonian $H = p\dot{q} - L$ associated with the resulting quadratic action (B.2). A general argument presented in appendix A shows that our gauge choice and properties of the Dirac-Born-Infeld action act together to guarantee that H takes the form

$$H = P^0 - |Q_{D0}|T_{D0} - |Q_{F1}|T_{F1}L_z\,. \tag{2.15}$$

We note in particular that the argument of appendix A applies equally well to fermionic and bosonic degrees of freedom.

One sees that (2.15) is not the energy of the system. Instead, our hamiltonian measures the extent to which a state is excited above the BPS bound (2.6).

The detailed results of the expansions in powers of fields are useful for the next section, but are not particularly enlightening in themselves. We will not burden the reader with such formulae here, reserving them instead for appendix B. We will, however, mention that in computing the quadratic action (B.2) from (2.1), we perform an integration by parts so that the canonical momenta take a more transparent form. We warn the reader in advance that this integration by parts performs a canonical transformation that causes the canonical momentum π_z (conjugate to a_z) below to differ by linear terms from the Π_z (2.3) defined from the original action (2.1). Thus, while the electric charge Q_{F1} remains the integral of Π_z, it is not the integral of the π_z used below.

3. The spectrum of states

We now use the results of section 2 to find the spectrum of states for our linearized system. In fact, we can simplify the treatment somewhat by realizing the momentum in the z direction breaks supersymmetry. Since we are interested in BPS states, we may thus restrict attention to modes independent of z. The action for such modes is given in (B.2), but the resulting equations of motion are:

$$\frac{R^2+B^2}{B}\partial_t^2 r + \mathrm{sgn}(E)2\left(\partial_t\partial_\sigma r - \frac{R}{B}\partial_t a_z\right) = 0 \tag{3.1}$$

$$\frac{R^2(R^2+B^2)}{B^3}\partial_t^2 a_z + \mathrm{sgn}(E)\left(\frac{2R^2}{B^2}\partial_t\partial_\sigma a_z + \frac{2R}{B}\partial_t r\right) = 0\,, \tag{3.2}$$

$$\frac{R^2+B^2}{B}\partial_t^2\eta^i + \mathrm{sgn}(E)2\partial_t\partial_\sigma\eta^i = 0\,, \qquad \text{and} \tag{3.3}$$

$$\frac{1}{B}\partial_t^2 a_\sigma = 0\,. \tag{3.4}$$

Note in particular that these equations are identically satisfied when all time derivatives vanish, so that all static configurations are allowed. We must also consider the Gauss Law constraint which due to gauge fixing no longer follows from our action. However, for z-independent modes in our gauge this is just $\partial_\sigma a_\sigma = 0$ at this order.

Without loss of generality, we choose $\mathrm{sgn}(E) = \mathrm{sgn}(B) = 1$. We first compute the relevant mode expansions in section (3.1) and then count the relevant states in section (3.2).

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3.1 Mode expansions

Each transverse degree of freedom η^i (for $i \in \{4, \dots, 9\}$) decouples from all other fields and has a solution of the form

$$\eta^i = \frac{1}{\sqrt{4\pi L_z T_{D2}}} \sum_{k_\sigma \neq 0} \frac{a^i_{k_\sigma}}{\sqrt{|k_\sigma|}} e^{i\omega_a t + ik_\sigma \sigma} + \frac{b^i_{k_\sigma}}{\sqrt{|k_\sigma|}} e^{ik_\sigma \sigma} , \tag{3.5}$$

where the normalizations have been chosen with foresight to simplify expressions to come. The relevant frequencies are

$$\omega_a(k_\sigma) = -\frac{2Bk_\sigma}{R^2 + B^2} , \qquad \text{and} \qquad \omega_b(k_\sigma) = 0 . \tag{3.6}$$

On the other hand, the radial and Maxwell degrees of freedom are coupled. Their solutions take the slightly more complicated form

$$\begin{aligned} r &= \frac{1}{2\sqrt{2\pi L_z T_{D2}}} \sum_{k_\sigma \neq \pm 1} \frac{a^\pm_{k_\sigma}}{\sqrt{|-k_\sigma \pm 1|}} e^{i\omega^\pm_a t + ik_\sigma \sigma} + \frac{b^\pm_{k_\sigma}}{\sqrt{|-k_\sigma \pm 1|}} e^{ik_\sigma \sigma} , &(3.7)\\ a_z &= \pm i \frac{B}{2R\sqrt{2\pi L_z T_{D2}}} \sum_{k_\sigma \neq \pm 1} \frac{a^\pm_{k_\sigma}}{\sqrt{|-k_\sigma \pm 1|}} e^{i\omega^\pm_a t + ik_\sigma \sigma} + \frac{b^\pm_{k_\sigma}}{\sqrt{|-k_\sigma \pm 1|}} e^{ik_\sigma \sigma} , &(3.8)\\ a_\sigma &= (\text{const}_1) t + \text{const}_2 , &(3.9) \end{aligned}$$

with the similar but slightly more complicated frequencies

$$\begin{aligned} \omega^\pm_a(k_\sigma) &= \frac{2B}{R^2 + B^2}(-k_\sigma \pm 1) , \qquad \text{and} &(3.10)\\ \omega^\pm_b(k_\sigma) &= 0 . &(3.11) \end{aligned}$$

The a_σ degree of freedom will not be of further interest below.

Note in particular that $\omega^\pm_a(k_\sigma)$ vanishes when $k = \pm 1$. These zero modes represent the translation symmetries in the X^1 and X^2 direction. After quantization, such modes become analogues of the free non-relativistic particle. The same is true of the η^i modes with $k_\sigma = 0$, associated with translations in X^i for $i \in \{4, \dots, 9\}$. A careful treatment shows that their velocities appear in the hamiltonian H, so that these modes are not annihilated by H even though they have zero frequency. In particular, these modes are not BPS. We will not concern ourselves with the detailed treatment of these zero modes here — the expressions below should be understood as correct only up to terms involving such modes.

In addition, we have $\omega^\pm_b(k_\sigma) = \omega_b(k_\sigma) = 0$ for all k_σ. This is just the linearized description of the known result [15] that the supertube allows arbitrary static deformations of its cross-section and magnetic field, so long as translation invariance in the z-direction is preserved. Although they have zero frequency, we will see below that such modes are *not* described by free particle degrees of freedom. Instead, the coefficients $a^\pm_{k_\sigma}$,$a^i_{k_\sigma}$ and $b^\pm_{k_\sigma}$,$b^i_{k_\sigma}$ are standard creation and annihilation operators which create or annihilate excitations of the round supertube. As a result, their vanishing frequency means that these modes *are* annihilated by the linearized Hamiltonain H. Since H encodes the BPS condition, it is

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clear that any $k_z = 0$ excitation of the b-modes preserves the BPS-bound. As a result, there are unbroken supersymmetry generators with which we may act to conclude that there are a similar set of 8 fermionic fields having static $k_z = 0$ excitations to fill out the BPS multiplet associated with 8 supercharges.

To se that the coefficients $a^\pm_{k_\sigma}$,$a^i_{k_\sigma}$ and $b^\pm_{k_\sigma}$,$b^i_{k_\sigma}$ are standard creation and annihilation operators, we note from the action (2.1) and the solutions (3.7), the canonical momenta π_z (conjugate to a_z) and $\mathcal{P}_r$, $\mathcal{P}_i$ take the form

$$\mathcal{P}_r = -\frac{i}{2}\sqrt{\frac{L_z T_{D2}}{2\pi}} \sum_{k_\sigma\neq\pm1} \frac{-k_\sigma \pm 1}{\sqrt{|-k_\sigma \pm 1|}} \left(a^\pm e^{i\omega_a^\pm t + ik_\sigma\sigma} - b^\pm e^{ik_\sigma\sigma}\right), \tag{3.12}$$

$$\pi_z = \pm\frac{R}{2B}\sqrt{\frac{L_z T_{D2}}{2\pi}} \sum_{k_\sigma\neq\pm1} \frac{-k_\sigma \pm 1}{\sqrt{|-k_\sigma \pm 1|}} \left(a^\pm e^{i\omega_a^\pm t + ik_\sigma\sigma} - b^\pm e^{ik_\sigma\sigma}\right), \tag{3.13}$$

$$\mathcal{P}_i = \sqrt{\frac{L_z T_{D2}}{4\pi}} \sum_{k_\sigma\neq 0} \frac{-k_\sigma}{\sqrt{|k_\sigma|}} \left(a^i e^{i\omega_a t + ik_\sigma\sigma} - b^i e^{ik_\sigma\sigma}\right). \tag{3.14}$$

A straightforward but lengthy calculation from the canonical commutation relations then shows that the a's and b's satisfy

$$[a^+_{k_\sigma}, a^-_{k'_\sigma}] = -\delta_{k_\sigma+k'_\sigma}\operatorname{sgn}(k_\sigma - 1)\,, \tag{3.15}$$

$$[b^+_{k_\sigma}, b^-_{k'_\sigma}] = \delta_{k_\sigma+k'_\sigma}\operatorname{sgn}(k_\sigma - 1)\,, \tag{3.16}$$

$$[a^i_{k_\sigma}, a^i_{k'_\sigma}] = -\delta_{k_\sigma+k'_\sigma}\operatorname{sgn}(k_\sigma)\,, \tag{3.17}$$

$$[b^i_{k_\sigma}, b^i_{k'_\sigma}] = \delta_{k_\sigma+k'_\sigma}\operatorname{sgn}(k_\sigma)\,, \tag{3.18}$$

while the remaining commutators vanish. In addition, the reality conditions require

$$(a^+_{k_\sigma})^\dagger = a^-_{-k_\sigma}\,, \qquad (b^+_{k_\sigma})^\dagger = b^-_{-k_\sigma}\,, \tag{3.19}$$

$$(a^i_{k_\sigma})^\dagger = a^i_{-k_\sigma}\,, \qquad (b^i_{k_\sigma})^\dagger = b^i_{-k_\sigma}\,. \tag{3.20}$$

Thus we may identify ($a^+_{k_\sigma}$,$b^-_{-k_\sigma}$) for $k_\sigma > 1$ and ($a^-_{-k_\sigma}$,$b^+_{k_\sigma}$) for $k_\sigma < 1$ as creation operators and their adjoints as annihilation operators. Similarly, ($a^i_{k_\sigma}$,$b^i_{-k_\sigma}$) for $k_\sigma > 0$ are the creation operators for the η-modes. In particular, for $k_z = 0$ the BPS (b) modes carry negative angular momentum around the cylinder while the non-BPS (a) modes carry positive angular momentum. This is in accord with the result of [15] that the round supertube is the unique BPS state of maximal angular momentum. As a result, the round state acts like a vacuum state relative to the set of BPS excitations.[3]

Finally, we wish to express the charges in terms of the creation and annihilation operators $a^\pm_{k_\sigma}$, $a^i_{k_\sigma}$ and $b^\pm_{k_\sigma}$, $b^i_{k_\sigma}$. Once again, the procedure is straightforward but lengthy. The resulting expressions are:

$$H = \sum_{k_\sigma>1} \omega_a^-(-k_\sigma) a^+_{k_\sigma} a^-_{-k_\sigma} + \sum_{k_\sigma<1} \omega_a^+(k_\sigma) a^-_{-k_\sigma} a^+_{k_\sigma} + \sum_{k_\sigma>0} \omega_a(k_\sigma) a^i_{k_\sigma} a^i_{-k_\sigma}\,, \tag{3.21}$$

[3]With the understanding that "excitations" *lower* the angular momentum instead of raising it.

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$$
\begin{aligned}
J &= J^{\text{round}} + \sqrt{2\pi L_z T_{D2}}\frac{R}{B}(b_0^+ + b_0^-) + \\
&\quad + \sum_{k_\sigma>1}\left[\frac{2R^2 k_\sigma}{R^2+B^2} + \frac{2B}{R^2+B^2} + \frac{1}{2(k_\sigma-1)}\right] a^+_{k_\sigma} a^-_{-k_\sigma} + \\
&\quad - \sum_{k_\sigma<1}\left[\frac{2R^2 k_\sigma}{R^2+B^2} + \frac{2B}{R^2+B^2} + \frac{1}{2(k_\sigma-1)}\right] a^-_{-k_\sigma} a^+_{k_\sigma} + \\
&\quad + \sum_{k_\sigma>1}\frac{b^-_{-k_\sigma} b^+_{k_\sigma}}{2(k_\sigma-1)} - \sum_{k_\sigma<1}\frac{b^+_{k_\sigma} b^-_{-k_\sigma}}{2(k_\sigma-1)} + \\
&\quad + \sum_{k_\sigma\neq\pm1}\frac{b^+_{k_\sigma} b^+_{-k_\sigma}}{4\sqrt{|1-k_\sigma^2|}} + \sum_{k_\sigma\neq\pm1}\frac{b^-_{k_\sigma} b^-_{-k_\sigma}}{4\sqrt{|1-k_\sigma^2|}} + \sum_{k_\sigma>0}\frac{2R^2 k_\sigma}{R^2+B^2} a^i_{k_\sigma} a^i_{-k_\sigma}\,, \qquad (3.22)\\
Q_{D0} &= Q_{D0}^{\text{round}} = \frac{2\pi L_z T_{D2}}{T_{D0}} B\,,\\
\Delta &:= Q_{F1} Q_{D0} - J\\
&= \sum_{k_\sigma>1} k_\sigma (b^-_{-k_\sigma} b^+_{k_\sigma} - a^+_{k_\sigma} a^-_{-k_\sigma}) - \sum_{k_\sigma<1} k_\sigma (b^+_{k_\sigma} b^-_{-k_\sigma} - a^-_{-k_\sigma} a^+_{k_\sigma}) + \\
&\quad + \sum_{k_\sigma>0} k_\sigma (b^i_{-k_\sigma} b^i_{k_\sigma} - a^i_{k_\sigma} a^i_{-k_\sigma}) \qquad (3.23)\\
&= P_\sigma^{\text{can}}\,. \qquad (3.24)
\end{aligned}
$$

Here we have chosen to emphasize the hamiltonian H instead of the energy P^0, though the latter is easily recovered through the relation (2.15). Since we have not explicitly included Fermions, normal ordering has been used to obtain a finite result for (3.21). We have also chosen to express the charge Q_{F1} in terms of Q_{D0} and the angular momentum, as one sees that the combination $\Delta = Q_{F1}Q_{D0} - J$ defined above takes a fairly simple form; it is just P_σ^{can}, the canonical generator of σ-translations in our gauge-fixed theory. A general argument for this form is given in appendix A which applies to both bosonic and Fermionic degrees of freedom, but we have also verified the result explicitly for the bosons. As forewarned, one may further check that Q_{F1} is not the integral of the electric field momentum π_z, even at the linear level.

3.2 Counting states

Let us now fix $H = 0$, Q_{D0}, and the quantity $\Delta := Q_{F1}Q_{D0} - J$ (but not Q_{F1} or J individually). We see from (3.23) that when restricted to BPS states (those with $\omega = 0$), the operator $Q_{F1}Q_{D0} - J$ takes the form of the energy of a system of 8 right-moving 1+1 massless scalars. Furthermore, the argument in appendix A shows that this follows from general considerations, and thus that the Fermionic contributions suppressed here must take the corresponding form. Thus, the entire system is a 1+1 right-moving CFT with 8 scalars and 8 Fermions, having central charge $c = 12$. Note that fixing Q_{D0} places no restrictions on such effective right-moving fields, as Q_{D0} is given by the magnetic flux, a topological invariant. Thus, the Cardy formula [22] tells us that the entropy of our system is $S = 2\pi\sqrt{2(Q_{D0}Q_{F1} - J)}$.

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What remains is to argue that the entropy depends on the charges *only* through the combination $Q_{D0}Q_{F1} - J$, and to tie up a loose end having to do with the quantization of charge and angular momentum. The latter issue arises from a careful inspection of (3.22), which shows that J (and thus Q_{F1}) has a linear term which necessarily leads to a continuous spectrum. That the spectrum of Q_{F1} is continuous is an artifact of our not yet imposing that the gauge group is compact.[4] To do so, we must quotient the configuration space of the connection by an appropriate translation. It turns out to be convenient to deal with both issues simultaneously.

To do so, let us recall that the above quotient compactify the configuration space of the zero mode $(a_z)_{k=0,k_z=0} = \frac{T_{F1}}{2\pi RL_zT_{D0}}\int dzd\sigma\ a_z$, where we have chosen the normalization to be such that $(a_z)_{k=0,k_z=0}$ is compactified with period 2π. Thus, while $(a_z)_{k=0,k_z=0}$ will no longer be a well-defined operator, the exponentiated operator $e^{in(a_z)_{k=0,k_z=0}}$ will be well-defined for any integer n.

It is useful to consider only the time independent part of this zero mode:

$$(a_z)_{k=0,k_z=0,\omega=0} := \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} dt(a_z)_{k=0,k_z=0}\,, \tag{3.25}$$

which depends only on the time independent (and BPS) b-modes. Note that the exponential $e^{in(a_z)_{k=0,k_z=0,\omega=0}}$ is again well-defined[5] for any integer n.

Now, since Π_z is the canonical conjugate to a_z defined by the action (2.1), conjugation of Q_{F1} by $e^{in(a_z)_{k=0,k_z=0,\omega=0}}$ will simply add n units of charge:

$$e^{-in(a_z)_{k=0,k_z=0,\omega=0}}Q_{F1}e^{in(a_z)_{k=0,k_z=0,\omega=0}} = Q_{F1} + n\,. \tag{3.26}$$

But we see explicitly that $e^{in(a_z)_{k=0,k_z=0,\omega=0}}$ commutes with the expression (3.23) for $Q_{D0}Q_{F1} - J$. Furthermore, since $e^{in(a_z)_{k=0,k_z=0,\omega=0}}$ is time-independent, it must commute with H and so maps BPS states to BPS states. There is thus a unitary (and, in particular, bijective) map acting within the class of BPS states that changes Q_{F1}, but leaves $Q_{D0}Q_{F1} - J$ invariant. It follows that the desired entropy can depend on the charges only through the combination $Q_{D0}Q_{F1} - J$ and thus that, when all charges are fixed, the entropy is indeed $2\pi\sqrt{2(Q_{D0}Q_{F1} - J)}$ to leading order in the charges.

3.3 Limits of validity

We have now attained our main goal and verified the conjectured form of the entropy within the domain of our linearized treatment. It is important, however, to characterize the size of this domain. After all, our use of Cardy's formula required $\Delta \gg 1$, and one might worry that this constraint might be in conflict with our linear treatment.

[4]The angular momentum J also has continuous spectrum, but it is a familiar result that quantization of J imposes the Dirac quantization condition on the product of electric and magnetic charge. Note that a proper description of magnetic charge again requires compactification of the gauge group.

[5]It is also gauge invariant. Invariance under small diffeomorphisms is manifest from the integration over the world-volume. Invariance under large diffeomorphisms may be checked, but in the end is essentially equivalent to the fact (3.26) that the operator translates Q_{F1} by an integer. We thank David Gross for raising this issue.

We need to estimate the size of some higher order correction to our calculations. However, since supertubes are exact solutions to the Born-Infeld action [8, 15], there are no corrections to the solutions at this level. Furthermore, it has been argued [26] that such supertube solutions receive no corrections from higher derivative terms in the D2 effective action.[6] Furthermore, the action vanishes when evaluated on supertube configurations. Thus, we will not obtain useful error estimates from the action or equations of motion.

On the other hand, our charges *do* receive corrections from the higher order terms: even for supertubes, the expression (2.3) is not quadratic. Thus we may estimate our errors by comparing contributions to Q_{F1} from different orders. Rather than calculate the third order term, we will simply compare the second-order contribution with the zero-order term. (Note that the linear term gives only a rather trivial shift of the background and, in particular, is independent of Δ.)

There are in fact two types of quadratic contributions to Q_{F1}: those appear in $\Delta = Q_{F1}Q_{D0} - J$ and those that appear in J. Restricting Δ to be small requires merely $\Delta \ll Q_{F1}Q_{D0}$.

Let us now consider the quadratic terms in J. We are interested only in the BPS modes, so we need only include those terms built from $b^{\pm}_{k_\sigma}$. Examination of (3.21) shows that typical matrix elements of such terms are of rough size $\sum_{k_\sigma \geq 1} N_{k_\sigma}/k_\sigma$, where N_{k_σ} is the number operator associated with each mode. Since k_σ takes values in the positive integers, such terms are always smaller than Δ and impose no further restriction.

4. Discussion

We have seen above that supertubes represent marginal bound states and that for $Q_{D0}Q_{F1} \gg Q_{D0}Q_{F1} - J \gg 1$ the entropy of supertube states is given to leading order by $S = 2\pi\sqrt{2(Q_{D0}Q_{F1} - J)}$. This is identical to the leading-order entropy of all such D0-F1 bound states. In particular, once the center-of-mass momenta are fixed we obtain a discrete set of supertube states despite the presence of an infinite number of zero-frequency modes. This result is perhaps most easily explained by noting [26] that σ is a null direction with respect to the (inverse) open-string metric (defined in [28]) on the supertube. Thus, our shape degrees of freedom are more similar to excitations of a 1+1 massless field than to those of the more familiar sort of zero mode. Note that it is in fact easier to count states in which all three of Q_{D0}, Q_{F1}, and J are fixed than when only Q_{F1}, Q_{D0} are fixed, since restricting $Q_{D0}Q_{F1} - J \ll Q_{F1}Q_{D0}$ allows us to treat the system perturbatively.

Our results support the conjecture that supertubes provide an effective description of generic D0-F1 bound states. It would be interesting to extend this analysis by applying similar techniques to the related supergravity solutions of [9] or [21], or perhaps by studying

[6]One may note that T-dualizing the $O(F^4)$ higher derivative terms obtained in [27] would appear to lead to such corrections. However, since $E = 1$ for the supertube one cannot expect the correct behavior to be obtained by considering corrections at any finite order in F. Thus [26] and [27] are not in conflict. We thank Iosef Bena for this observation.

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the linearization around other (less symmetric) Born-Infeld supertube configurations. In addition, it would be of interest to relate our entropy calculations to the entropy of the two-charge black rings of Emparan and Elvang [29].

We note that results for the multiply wound case where $\tan(X^1/X^2) = \sigma/n$ may also be of interest. Such results are easily obtained from those above by applying the methods of appendix A and noting that the only change is the replacement $J \to nJ$ as the tube now rotates n times in the X^1X^2 plane under $\sigma \to \sigma + 2\pi$. Thus, the entropy of small fluctuations about the round tube with n wrappings is given by $S = 2\pi\sqrt{2(Q_{D0}Q_{F1} - nJ)}$. For fixed Q_{D0}, Q_{F1}, J we see that the entropy is greatest for the case $n = 1$.

The results above are of use for understanding the two-charge system, but similar studies for the related three-charge systems could have implications for black holes and thus be of much greater interest. In particular, Mathur et al. [11]–[14] have conjectured that similar results hold for such three-charge systems: that almost all such bound states can be described in terms of extended horizon-free configurations in which the entropy is readily apparent, for example with the distinct states being labelled by the shape of the object and the values of associated worldvolume fields. If this were so, it would leave no room for black holes as a distinct class of states. Thus, Mathur et al wish to conjecture that black holes represent only an effective statistical average over collections of more fundamental states; see [11]–[13] for details.

Such conjectures cannot be studied using our abelian D2 effective action, as the third charge in this context corresponds to adding D4-branes orthogonal to the tube directions. However, one may imagine studying linearizations of the three-charge Dirac-Born-Infeld system of [24]. Such calculations are currently underway in joint work with the authors of [24].

Perhaps even more interesting would be to study in detail linearized fluctuations of the known smooth [21] D1-D5 supergravity solutions.[7] However, merely adding momentum to such solutions seems unlikely to yield enough states to account for the full entropy of the three-charge system. In particular, in order to find enough entropy it is not sufficient simply to find BPS modes with arbitrarily k_z. Instead, recall that the 3-charge entropy involves the product $Q_{D1}Q_{D5}P$, whereas simply adding a $1+1$ field theory in the $z-t$ plane would at most add a function of P to the 2-charge entropy. Thus, it would appear that one would need to find BPS modes with k_σ and k_z both arbitrary. In other words, the tube must not only support travelling waves, but must support independent travelling waves at each value of σ. Such a result cannot be obtained from any non-degenerate quadratic 2+1 dispersion relation, though we cannot immediately rule out the possibility that it might arise due to complicated linear couplings between the various degrees of freedom. We understand that related issues are currently being explored by Mathur and collaborators (in an extension of [14]). Working from a different perspective, we also hope to report related further results (with our collaborators) in the near

[7]Related studies of metrics with conical deficits were begun in [14] and are being continued by those authors.

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future. Finally, it remains possible that other yet-unsuspected generalizations of known solutions will lead to enough states to account for the full entropy of the three-charge system.

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A. Gauge fixing and charges

In this appendix we show how the important relations (2.15) and (3.23) follow directly from general considerations of symmetries and our gauge fixing scheme. As a result, they represent a useful check on our bosonic calculations. In addition, we note that the argument below applies equally well to the Fermionic fields which we have not treated explicitly.

It will be helpful to distinguish here between the full Dirac-Born-Infeld lagrangian of (2.1), which we denote L, and the quadratic gauge fixed lagrangian ($L^{(2)}_{gf}$) explicitly displayed in (B.2). We remind the reader that $L^{(2)}_{gf}$ is obtained from L in two stages, first gauge fixing L to form L_{gf}, and then taking the quadratic term which yields $L^{(2)}_{gf}$. In particular, note that passing to $L^{(2)}_{gf}$ discards the constant term corresponding to evaluating L on our background, as this term is of order zero in our perturbations.

In fact, we argue in somewhat more generality below. Let us consider the lagrangian $\tilde{L}_{gf}$ which differs from L_{gf} only by subtracting the background value, while retaining all of the higher terms:

$$\tilde{L}_{gf} := L_{gf} - L|_{\text{Background}} = L^{(2)}_{gf} + \text{ higher order terms}\,. \tag{A.1}$$

We begin by noting that invariance under t and σ reparametrizations implies two important identities for L, which we may call the hamiltonian and momentum constraints:

$$\sum_\mu \frac{\partial L}{\partial(\partial_t X^\mu)}\partial_t X^\mu + \sum_i \frac{\partial L}{\partial(\partial_t A_i)}\partial_t A_i = L \tag{A.2}$$

$$\sum_\mu \frac{\partial L}{\partial(\partial_t X^\mu)}\partial_\sigma X^\mu + \sum_i \frac{\partial L}{\partial(\partial_t A_i)}\partial_\sigma A_i = 2\sum_i \frac{\partial L}{\partial F_{0i}}\partial_i A_\sigma . \tag{A.3}$$

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We now use the first of these results to identify H in terms of P^0, Q_{F1} and Q_{D0}. The hamiltonian H is by definition

$$\begin{aligned} H &= \int dz d\sigma \left(\frac{\partial \tilde{L}_{gf}}{\partial(\partial_t \eta^\mu)} \partial_t \eta^\mu + \sum_i \frac{\partial \tilde{L}_{gf}}{\partial(\partial_t a_i)} \partial_t a_i - \tilde{L}_{gf} \right) \\ &= \int dz d\sigma \left(\sum_\mu \frac{\partial L_{gf}}{\partial(\partial_t X^\mu)} \partial_t X^\mu + \sum_i \frac{\partial L_{gf}}{\partial(\partial_t A_i)} \partial_t A_i - \mathrm{sgn}(E)\Pi_z - L_{gf} + L|_{\mathrm{Background}} \right), \end{aligned}$$

where we have used (A.1) and the fact that the only time-dependent background field not completely fixed by the gauge condition is A_z, whose time derivative is $E = \pm 1$ and whose conjugate momentum defined from (2.1) is Π_z. Recall that canonical transformations do not affect the hamiltonian, so that we may consistently ignore the extra integration by parts mentioned below (B.2) which would replace Π_z by π_z and perform a compensating change in L_{gf}.

Now, L_{gf} is obtained from L by imposing the requirements $X^0 = t, X^1 = \sigma, X^2 = z$ and $A_0 = 0$. We denote this process by $|_{gf}$, e.g. $L_{gf} = L|_{gf}$. Expressing H in terms of L, we find

$$\begin{aligned} H &= \int dz d\sigma \Big(\sum_\mu \frac{\partial L}{\partial(\partial_t X^\mu)} \partial_t X^\mu - \frac{\partial L}{\partial(\partial_t X^0)} + \sum_i \frac{\partial L}{\partial(\partial_t A_i)} \partial_t A_i - \\ &\qquad - \mathrm{sgn}(E)\Pi_z - L + L|_{\mathrm{Background}} \Big)|_{gf}, \\ &= -\int dz d\sigma \left(\frac{\partial L}{\partial(\partial_t X^0)} + \mathrm{sgn}(E)\Pi_z - L|_{\mathrm{Background}} \right)|_{gf}, \end{aligned} \tag{A.4}$$

where in the last line we have used the hamiltonian constraint (A.2).

Finally, the general form of the Dirac-Born-Infeld action implies the relation

$$\frac{\partial L}{\partial G_{00}}|_{G=\eta} = -\frac{1}{2} \sum_a \frac{\partial L}{\partial(\partial_a X^0)} \partial_a X^0, \tag{A.5}$$

where $|_{G=\eta}$ denotes that we evaluate the expression (after taking the derivative) for the special case where G_{ab} is the Minkowski metric. After gauge fixing this becomes

$$T_{00}|_{gf,G=\eta} = 2 \frac{\partial L}{\partial G_{00}} \Big|_{gf,G=\eta} = -\frac{\partial L}{\partial(\partial_t X^0)}|_{gf,G=\eta}. \tag{A.6}$$

Using this together with $L_{\mathrm{Background}} = -B\mathrm{sgn}(B)$ and the definition of the charges (2.2), (2.3) we find

$$\begin{aligned} H &= \int dz d\sigma \left(T_{00} - \mathrm{sgn}(E)\Pi_z - \mathrm{sgn}(B) T_{D2} B \right)|_{gf} \\ &= P^0 - |Q_{F1}| L_z T_{F1} - |Q_{D0}| T_{D0}, \end{aligned} \tag{A.7}$$

where in the final step we have used the fact that the integrated magnetic flux is a topological invariant and so is always given by its value in the round tube background. Again

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we emphasize that the validity of (A.7) is in no way restricted to the linear approximation. Note that the main text primarily studies the case $\mathrm{sgn}(E) = \mathrm{sgn}(B) = 1$ for which $Q_{F1}, Q_{D0} > 0$.

Now, H is the generator of time translations in the quadratic gauge-fixed theory. We may of course also consider P_σ^{can}, the generator of σ-translations in the quadratic gauge-fixed theory. Applying the analogous reasoning to P_σ^{can} (but using the momentum constraint A.3), one arrives at the relation

$$P_\sigma^{\mathrm{can}} = Q_{F1} Q_{D0} - J\,. \tag{A.8}$$

We note that the form of (3.23) then follows immediately from our identification of the creation and annihilation operators.

B. Quadratic expansions

This appendix simply lists the formulae, suppressed in the main text, which describe 1) the quadratic expansions of the action for the z-independent fields and 2) the charges in terms of the perturbations η^μ, a_i.

The action for the z-independent modes is

$$\begin{aligned}
S &= S_{\mathrm{round}} + S^{(2)} + \text{ higher order terms} \qquad\qquad (\mathrm{B.1})\\
S^{(2)} &= -\mathrm{sgn}(B)\frac{L_z T_{D2}}{2}\int dt d\sigma\Big[-\frac{R^2+B^2}{B}((\partial_t r)^2+|\partial_t\eta|^2) - 2\mathrm{sgn}(E)(\partial_t r\partial_\sigma r+\partial_t\eta^i\partial_\sigma\eta^i) -\\
&\qquad - \mathrm{sgn}(E)\frac{2R}{B}(\partial_t a_z r - \partial_t r a_z) - \frac{R^2(R^2+B_0^2)}{B^3}(\partial_t a_z)^2 -\\
&\qquad - \frac{1}{B}(\partial_t a_\sigma)^2 - \mathrm{sgn}(E)\frac{2R^2}{B^2}\partial_t a_z\partial_\sigma a_z\Big]\,. \qquad\qquad (\mathrm{B.2})
\end{aligned}$$

In computing (B.2) from (2.1) we have performed an integration by parts, which induces a canonical transformation designed to make the momenta (3.12) take a more symmetric form. As a result, the canonical momentum π_z, defined by the action (B.2) conjugate to the connection to differ by linear terms from the Π_z (2.3), defined by (2.1). Thus, while the electric charge Q_{F1} remains the integral of Π_z, it is not the integral of π_z.

The charges and hamiltonian take the form

$$\begin{aligned}
Q_{D0} &= Q_{D0}^{\mathrm{round}}\,, \qquad\qquad (\mathrm{B.3})\\
Q_{F1} &= \frac{1}{T_{F1}}\int d\sigma \Pi_z\\
&= Q_{F1}^{\mathrm{round}} + \mathrm{sgn}(EB)\frac{T_{D2}}{2T_{F1}}\times\\
&\quad\times\int d\sigma\Big[\frac{4R}{B}r + 2\mathrm{sgn}(E)\frac{R^2(R^2+B^2)}{B^3}e_z +\\
&\qquad + \frac{R^2(R^2+B^2)}{B^3}((\partial_t r)^2+|\partial_t\eta|^2) - \frac{R^4}{B^3}((\partial_z r)^2+|\partial_z\eta|^2) +\\
&\qquad + \frac{2}{B}(r^2+(\partial_\sigma r)^2+|\partial_\sigma\eta|^2) + \mathrm{sgn}(E)\frac{2(R^2+B^2)}{B^2}(\partial_t r\partial_\sigma r+\partial_t\eta^i\partial_\sigma\eta^i) +
\end{aligned}$$

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$$
\begin{aligned}
&+\frac{3R^4(R^2+B^2)}{B^5}e_z^2+\frac{2R^2}{B^3}b^2+\frac{R^2}{B^3}e_\sigma^2-\\
&-\operatorname{sgn}(E)\frac{2R^2(3R^2+B^2)}{B^4}e_zb-\frac{4R}{B^2}rb+\operatorname{sgn}(E)\frac{4R(2R^2+B^2)}{B^3}re_z\bigg], \qquad \text{(B.4)}
\end{aligned}
$$

$$
\begin{aligned}
J =\; & J^{\text{round}}+\operatorname{sgn}(B)\frac{T_{D2}}{2}\int dzd\sigma\bigg[\operatorname{sgn}(E)4Rr+2\frac{R^2(R^2+B^2)}{B^2}e_z+\operatorname{sgn}(E)2r^2+\\
&+\operatorname{sgn}(E)\frac{R^2(R^2+B^2)}{B^2}((\partial_t r)^2+|\partial_t\eta|^2)-\\
&-\frac{R^4}{B^2}((\partial_z r)^2+|\partial_z\eta|^2)+\operatorname{sgn}(E)\frac{3R^4(R^2+B^2)}{B^4}e_z^2+\\
&+\frac{R^4}{B^2}e_\sigma^2-\frac{4R^4}{B^3}e_zb+\frac{4R(2R^2+B^2)}{B^2}re_z\bigg], \qquad \text{(B.5)}
\end{aligned}
$$

$$
\begin{aligned}
P^0 =\; & P^0_{\text{round}}+\frac{\operatorname{sgn}(B)}{2}T_{D2}\times\\
&\times\int dzd\sigma\bigg[\frac{4R}{B}r+2\operatorname{sgn}(E)\frac{R^2(R^2+B^2)}{B^3}e_z+\\
&+\frac{(R^2+B^2)^2}{B^3}((\partial_t r)^2+|\partial_t\eta|^2)-\frac{R^2(R^2-B^2)}{B^3}((\partial_z r)^2+|\partial_z\eta|^2)+\\
&+\frac{2}{B}(r^2+(\partial_\sigma r)^2+|\partial_\sigma\eta|^2)+\operatorname{sgn}(E)\frac{2(R^2+B^2)}{B^2}(\partial_t r\partial_\sigma r+\partial_t\eta^i\partial_\sigma\eta^i)+\\
&+\frac{R^2(3R^2+B^2)(R^2+B^2)}{B^5}e_z^2+\frac{2R^2}{B^3}b^2+\frac{R^2+B^2}{B^3}e_\sigma^2- \qquad \text{(B.6)}\\
&-\operatorname{sgn}(E)\frac{2R^2(3R^2+B^2)}{B^4}e_zb-\frac{4R}{B^2}rb+\operatorname{sgn}(E)\frac{4R(2R^2+B^2)}{B^3}re_z\bigg],
\end{aligned}
$$

$$
\begin{aligned}
H =\; & P^0-|Q_{D0}|T_{D0}-|Q_{F1}|T_{F1}L_z\,,\\
=\; & \operatorname{sgn}(B)\frac{T_{D2}}{2}\int dzd\sigma\;\bigg[\frac{R^2+B^2}{B}((\partial_t r)^2+|\partial_t\eta|^2)+\frac{R^2}{B}((\partial_z r)^2+|\partial_z\eta|^2)+\\
&+\frac{R^2(R^2+B^2)}{B^3}(\partial_t a_z)^2+\frac{1}{B}(\partial_t a_\sigma)^2\bigg]. \qquad \text{(B.7)}
\end{aligned}
$$

Note in particular that H is not the energy P^0 that couples to the gravitational field. Instead, H measures the extent to which a state is excited above the BPS bound. Note also that expressions (B.3)–(B.7) are valid even when the fields depend on z.

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Erratum

The original version of appendix A contains several errors. For the readers convenience, the full (corrected) appendix is presented here.

In this appendix we show how the important relations (2.15) and (3.23) follow directly from general considerations of symmetries and our gauge fixing scheme. As a result, they represent a useful check on our calculations.

It will be helpful to distinguish here between the full Dirac-Born-Infeld lagrangian of (2.1), which we denote L, and the quadratic gauge fixed lagrangian ($L^{(2)}_{gf}$) explicitly displayed in (B.2). We remind the reader that $L^{(2)}_{gf}$ is obtained from L in two stages, first gauge fixing L to form L_{gf}, and then taking the quadratic term which yields $L^{(2)}_{gf}$. In particular, note that passing to $L^{(2)}_{gf}$ discards the constant term corresponding to evaluating L on our background, as this term is of order zero in our perturbations.

In fact, we argue in somewhat more generality below. Let us consider the lagrangian $\tilde{L}_{gf}$ which differs from L_{gf} only by subtracting the background value, while retaining all of the higher terms:

$$\tilde{L}_{gf} := L_{gf} - L|_{\text{Background}} = L^{(2)}_{gf} + \text{higher order terms}\,. \tag{B.8}$$

We begin by noting that invariance under t and σ reparametrizations implies two important identities for L, which we may call the hamiltonian and momentum constraints:

$$\sum_\mu \frac{\partial L}{\partial(\partial_t X^\mu)}\partial_t X^\mu + \sum_i \frac{\partial L}{\partial(\partial_t A_i)} F_{ti} = L \tag{B.9}$$

$$\sum_\mu \frac{\partial L}{\partial(\partial_t X^\mu)}\partial_\sigma X^\mu + \sum_i \frac{\partial L}{\partial(\partial_t A_i)} F_{\sigma i} = 0\,. \tag{B.10}$$

We now use the first of these results to identify H in terms of P^0, Q_{F1} and Q_{D0}. The hamiltonian H is by definition

$$\begin{aligned} H &= \int dz d\sigma \left(\frac{\partial \tilde{L}_{gf}}{\partial(\partial_t \eta^\mu)}\partial_t\eta^\mu + \sum_i \frac{\partial \tilde{L}_{gf}}{\partial(\partial_t a_i)}\partial_t a_i - \tilde{L}_{gf} \right) \\ &= \int dz d\sigma \left(\sum_\mu \frac{\partial L_{gf}}{\partial(\partial_t X^\mu)}\partial_t X^\mu + \sum_i \frac{\partial L_{gf}}{\partial(\partial_t A_i)}\partial_t A_i - \text{sgn}(E)\Pi_z - L_{gf} + L|_{\text{Background}} \right), \end{aligned}$$

where we have used (B.8) and the fact that the only time-dependent background field not completely fixed by the gauge condition is A_z, whose time derivative is $E = \pm 1$ and whose conjugate momentum defined from (2.1) is Π_z. Recall that canonical transformations do not affect the hamiltonian, so that we may consistently ignore the extra integration by parts mentioned below (B.2) which would replace Π_z by π_z and perform a compensating change in L_{gf}.

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Now, L_{gf} is obtained from L by imposing the requirements $X^0 = t$, $X^3 = z$, $X^1 = R(t,z,\sigma)\cos\sigma$, $X^2 = R(t,z,\sigma)\sin\sigma$ and $A_0 = 0$. We denote this process by $|_{gf}$, e.g. $L_{gf} = L|_{gf}$. Expressing H in terms of L, we find

$$\begin{aligned} H &= \int dz d\sigma \Bigg(\sum_\mu \frac{\partial L}{\partial(\partial_t X^\mu)}\partial_t X^\mu - \frac{\partial L}{\partial(\partial_t X^0)} + \sum_i \frac{\partial L}{\partial(\partial_t A_i)}\partial_t A_i - \\ &\qquad - \operatorname{sgn}(E)\Pi_z - L + L|_{\text{Background}} \Bigg)\Bigg|_{gf} , \\ &= -\int dz d\sigma \left(\frac{\partial L}{\partial(\partial_t X^0)} + \operatorname{sgn}(E)\Pi_z - L|_{\text{Background}} \right)\Bigg|_{gf} , \end{aligned} \tag{B.11}$$

where in the last line we have used the hamiltonian constraint (B.9).

Finally, the general form of the Dirac-Born-Infeld action implies the relation

$$\left.\frac{\partial L}{\partial G_{00}}\right|_{G=\eta} = -\frac{1}{2}\sum_a \frac{\partial L}{\partial(\partial_a X^0)}\partial_a X^0 , \tag{B.12}$$

where $|_{G=\eta}$ denotes that we evaluate the expression (after taking the derivative) for the special case where G_{ab} is the Minkowski metric. After gauge fixing this becomes

$$T_{00}|_{gf,G=\eta} = 2\left.\frac{\partial L}{\partial G_{00}}\right|_{gf,G=\eta} = -\left.\frac{\partial L}{\partial(\partial_t X^0)}\right|_{gf,G=\eta} . \tag{B.13}$$

Using this together with $L_{\text{Background}} = -B\operatorname{sgn}(B)$ and the definition of the charges (2.2), (2.3) we find

$$\begin{aligned} H &= \int dz d\sigma \left(T_{00} - \operatorname{sgn}(E)\Pi_z - B\operatorname{sgn}(B) \right)|_{gf} \\ &= P^0 - |Q_{F1}|L_z T_{F1} - |Q_{D0}|T_{D0}, \end{aligned} \tag{B.14}$$

where in the final step we have used the fact that the integrated magnetic flux is a topological invariant and so is always given by its value in the round tube background. Again we emphasize that the validity of (B.14) is in no way restricted to the linear approximation. Note that the main text primarily studies the case $\operatorname{sgn}(E) = \operatorname{sgn}(B) = 1$ for which $Q_{F1}, Q_{D0} > 0$.

Now, H is the generator of time translations in the quadratic gauge-fixed theory. We may of course also consider P_σ^{can}, the generator of σ-translations in the quadratic gauge-fixed theory. We apply the analogous reasoning to P_σ^{can}, which by definition takes the form

$$\begin{aligned} P_\sigma &= \int dz d\sigma \frac{\partial \tilde{L}_{gf}}{\partial(\partial_t \eta^\mu)}\partial_\sigma \eta^\mu + \frac{\partial \tilde{L}_{gf}}{\partial(\partial_t a_i)}(\partial_\sigma a_i - \partial_i a_\sigma) \\ &= \int dz d\sigma \sum_{i=3}^{8} \frac{\partial L_{gf}}{\partial(\partial_t X^i)}\partial_\sigma X^i + \frac{\partial L_{gf}}{\partial(\partial_t R)}\partial_\sigma R + \frac{\partial L_{gf}}{\partial(\partial_t A_z)}(\partial_\sigma A_z - \partial_z A_\sigma + B) . \end{aligned} \tag{B.15}$$

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Let us now compute,

$$\frac{\partial L}{\partial(\partial_t X^1)}\partial_\sigma X^1\bigg|_{gf}+\frac{\partial L}{\partial(\partial_t X^2)}\partial_\sigma X^2\bigg|_{gf}=\partial_\sigma R\left(\cos\sigma\frac{\partial L}{\partial(\partial_t X^1)}\bigg|_{gf}+\sin\sigma\frac{\partial L}{\partial(\partial_t X^2)}\bigg|_{gf}\right)+$$
$$+\left(R\cos\sigma\frac{\partial L}{\partial(\partial_t X^2)}\bigg|_{gf}-R\sin\sigma\frac{\partial L}{\partial(\partial_t X^1)}\bigg|_{gf}\right)$$
$$=\partial_\sigma R\frac{\partial L_{gf}}{\partial(\partial_t R)}+\mathcal{L}_{12}\,, \tag{B.16}$$

where $\mathcal{L}_{12}$ is the density along the S^1 of the component of angular momentum (which we have called J) associated with the X^1X^2 plane. Substituting the above expression in (B.15),

$$P_\sigma=\int dzd\sigma\left(\frac{\partial L}{\partial(\partial_t X^\mu)}\partial_\sigma X^\mu+\frac{\partial L}{\partial(\partial_t A_z)}F_{\sigma z}\right)\bigg|_{gf}+\Pi_z B-\mathcal{L}_{12}\,, \tag{B.17}$$

and using the identity (B.10), one arrives at the relation

$$P_\sigma^{\rm can}=Q_{F1}Q_{D0}-J\,. \tag{B.18}$$

We note that the form of (3.23) then follows immediately from our identification of the creation and annihilation operators.

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